COMPACTNESS AND RELATED CONCEPTS

A THESIS

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INTRODUCTION

The purpose of this study is to investigate the basic properties of the mathematical concept of compactness in a topological space, together with several concepts which are related to compactness. A reading of this exposition will require a basic knowledge of elementary topology. The notions of a topological space, open and closed sets, bases for topologies, accumulation points, closures and interiors of sets, connectedness, continuous functions, and set theory are used freely and without explanation. The concept of compactness in the modern sense had its beginning in the classic real number theorem of Heine, Borel, and Lebesgue. This theorem (which is proved on page 19) states that a set of real numbers is closed and bounded if and only if it has the property that every open covering of the set has a finite subcovering. It is this property, which is the conclusion of the theorem, that is chosen by the topologist for generalization. Another important classical result, the Bolzano-Weierstrass theorem, yields another generalization of the same idea. This theorem states that a set of real numbers is closed and bounded if and only if every infinite subset has at least one accumulation point in the set. The property which is the conclusion of this theorem has also been generalized in a topological setting. It is called B-compactness in this paper.

The first part of Chapter One consists of a study of properties which are equivalent to compactness in spaces with certain prescribed structure. In particular, the concepts of countable compactness, B-compactness, and sequential compactness are investigated. Some confusion arises when one reads literature in which any of the above concepts appear. The term "countable compactness," as defined in this paper, is used in the same sense by J. L. Kelley in [2]. However, many authors, including [1] and [3], use the name countable compactness to mean what is meant in this paper by the term "B-compactness." These authors give no name to the other concept. The term "B-compactness" was invented by this writer; and is meant to suggest "Bolzano compactness," in allusion to the Bolzano-Weierstrass theorem. Historically the original meaning for the term "compact" was that of B-compactness. The present definition of compact, sometimes known as the "Heine-Borel property," was called "Bi-compact." Finally, in literature in which all spaces involved are metric, authors may define the term "compact" to mean any one of the four concepts mentioned above. Theorem I-20 proves the equivalence of these definitions for a metric space. The diagram in Figure 1 on page 11 summarizes the equivalences of these four terms under proper hypothesis on the space.

Section D of Chapter One treats compactness in a Hausdorff space to the extent that will be needed in Chapter Two. This treatment is by no means exhaustive. Sections C and E are also of this nature. The Tychonoff theorem is included without proof in Section G in order to present Example I-4. This theorem is one of the most important results of topology; however, no other use is made of it in this paper. Section H is a detailed investigation of continuous transformations of compact spaces. The problem of invariance of local compactness is also treated in Chapter Two in the light of compact mappings.

Chapter Two is an introduction to one of many generalizations of the idea of a compact space. It was found that by placing a condition of compactness on the transformation, rather than on the domain space, many useful results could be generalized. The notion of a compact mapping is relatively new and is currently an object of intensive research. The original concept was invented by the Russian mathematician, I. A. Vaĭnšteĭn, but his definition differs from the one given here. The definition of a compact mapping given in this paper is that of G. T. Whyburn. The equivalence of the two definitions is given in Corollary II-6 and Corollary II-8. The most important result of Chapter Two is the monotone-light factorization theorem (Theorem II-23) of Whyburn and S. Eilenburg. Originally it was proved for the case of a compact domain space and later extended for compact mappings. The theorem has numerous applications in topology and analysis.

Most of the results of Chapter Two may be found in [4] and [5]. The source of the theorems for each section of Chapter Two is given at the end of the section. The basic results of Section D of Chapter Two on compact mappings and homeomorphisms were communicated to this writer by Dr. Robert H. Kasriel. It should be noted that the basic assumption of a separable metric space by Whyburn in [4] is dropped from the theorems taken from that reference. The countability and separation hypotheses are substituted as needed, so that the proofs remain virtually the same. The reader unfamiliar with these various countability and separation properties may consider all spaces separable and metric without losing significant generality. An alternate concept, the sequentially compact mapping, may be defined by substituting the words "sequentially compact" for the word "compact" in Definition II-1. It can be seen that many of the theorems of Chapter Two may be stated with less hypothesis using this alternate concept. The role of sequential compactness is evident in the proofs.

Compactness has been called one of the most far-reaching concepts of topology. This thesis is a circumstantial introduction to compactness and a few of its applications in topology. Many other generalizations and applications have been made, not only in topology, but also throughout mathematics.

CHAPTER I

COMPACTNESS

A. Compact Spaces

<u>Definition I-1</u>. A topological space is said to be compact if every open covering of the space contains a finite subcovering.

<u>Remark</u>. The concept of a topological space used in this thesis is that which may be found on page 52 of [1].

<u>Definition I-2</u>. A collection of subsets of a topological space is said to have the finite intersection property if every finite subcollection has a nonempty intersection.

<u>Theorem I-1</u>. A topological space X is compact if and only if every collection of closed subsets of X having the finite intersection property has a nonempty intersection.

<u>Proof</u>. (1) Assume that X is compact, and let $\{C_a : a \in A\}$ be a collection of closed sets having the finite intersection property. Define $O_a = X - C_a$ for each a in A, and assume that the intersection of all elements of $\{C_a\}$ is empty. Then the collection $\{O_a : a \in A\}$ is an open covering of X. Since X is compact, a finite subcovering $\{O_{a_i} : i = 1, 2, \ldots, n\}$ covers X. Hence it follows that $\bigcap_{i=1}^{n} C_{a_i}$ is empty. This is a contradiction to the finite intersection property. Therefore $\bigcap_{a\in A} C_a$ is nonempty.

(2) Now assume that X has the property that every collection of closed subsets of X having the finite intersection property has a nonvoid intersection. Let $\{O_a : a \in A\}$ be an open covering of X. Assume no finite subcovering of $\{O_a\}$ covers X. Then the collection $\{C_a : C_a = X - O_a, a \in A\}$ of closed subsets of X has the finite intersection property and consequently has a nonvoid intersection. Therefore the collection $\{O_a\}$ fails to cover X. This is a contradiction.

<u>Definition I-3</u>. A set D is said to be directed by the relation $\geq$ if D is nonvoid and

(a) if m, n, and p are members of D such that $m \geq n$ and $n \geq p$, then $m \geq p$;

(b) if $m \in D$, then $m \geq m$; and

(c) if m and n are members of D, then there is an element p in D such that $p \geq m$ and $p \geq n$.

<u>Definition I-4</u>. A net is a pair $(S, \geq)$ where S is a function and $\geq$ directs the domain of S. The notation $\{S_n, n \in D\}$, where S_n are values in a topological space X and D is a directed set, will be used to denote a net in a space X.

<u>Definition I-5</u>. A net $\{T_i, i \in E\}$ is a subnet of a net $\{S_n, n \in D\}$ if

(a) for each i in E, $T_i = S_{n_i}$, $n_i \in D$;

(b) for each m in D there is a q in E with the property that, if $p \geq q$, then $n_p \geq m$.

<u>Theorem I-2</u>. A topological space X is compact if and only if every net whose range is in X has a subnet which converges to a point in X.

<u>Proof</u>. Suppose first that X is compact. Let $\{S_n, n \varepsilon D\}$ be a net in X. The sets $A_n = \{S_m : m \geq n\}$ have the finite intersection property because D is directed by $\geq$. Therefore the collection of closures $\{\overline{A}_n\}$ also has the finite intersection property. Hence $\bigcap_{n\varepsilon D} \overline{A}_n$ is nonempty. Let $s \varepsilon \bigcap_{n\varepsilon D} \overline{A}_n$ and let γ be a neighborhood system about s. Note that γ is directed by set inclusion. Construct a subnet $\{T_i : i \varepsilon E\}$ in the following manner. Let E be the set of all ordered pairs (n, U) where $n \varepsilon D$, $U \varepsilon \gamma$, and $S_n \varepsilon U$. Let $\geq$ be an ordering for E defined by: $(O, m) \geq (U, n)$ if $m \geq n$ and $U \supset O$. Then E together with $\geq$ is a directed set, since for each (n, U_1) and (m, U_2) there exists $U \varepsilon \gamma$ such that $U \subset U_1 \cap U_2$ and an element $p \varepsilon D$ such that $p \geq n$, $p \geq m$, and $S_p \varepsilon U$. Then $(p, U) \geq (n, U_1)$ and $(p, U) \geq (m, U_2)$. Define the function $h : E \rightarrow D$ by $h[(n, U)] = n$. It must be shown that $\{T_i : i \varepsilon E\}$ with $T_i = S_{h(i)}$ is a subnet of $\{S_n : n \varepsilon D\}$ which converges to s. Clearly property (a) of Definition I-5 is satisfied. For each m in D there exists an $n \varepsilon D$ such that $n \geq m$ and $S_n \varepsilon U$, where $U \varepsilon \gamma$. Let $q = (n, U)$. Then for any $p = (n', U')$ such that $p \geq q$, $h(p) = n_p = n' \geq m$, since $n' \geq n$. Hence property (b) of Definition I-5 is satisfied. Therefore $\{T_i : i \varepsilon E\}$ is a subnet of $\{S_n : n \varepsilon D\}$. Clearly $\{T_i : i \varepsilon E\}$ converges to s.

(2) Now assume that every net in X has a convergent subnet. Let α be a collection of closed subsets of X with the finite intersection property. Define the collection β to be the collection

of all sets which are finite intersections of sets of α. Then β also has the finite intersection property and $\beta \supset \alpha$. It will be shown that $\bigcap_{B\varepsilon\beta} B$ is nonempty. Clearly β is directed by set inclusion. Let S_B be a member of B for each $B \varepsilon \beta$. Then $\{S_B : B \varepsilon \beta\}$ is a net in X and consequently has a subnet $\{S_{B_i} : i \varepsilon E\}$ which converges to an element s in S. Let B be an arbitrary element of β, and let U be a neighborhood of a. It will be shown that $s \varepsilon B$. For some $i \varepsilon E$, $S_{B_m} \varepsilon U$ for every $m \geq i$. Furthermore there is a q in E with the property that, if $p \geq q$, then $B_q \subset B$. Let r be an element of E succeeding both i and q. Then $S_{B_r} \varepsilon U$ and $B_r \subset B$. But $S_{B_r} \varepsilon B_r$; hence $S_{B_r} \varepsilon B$. Consequently s belongs to B, since B is closed. It follows that $s \varepsilon \bigcap_{B\varepsilon\beta} B$.

B. Properties of Compact Spaces

Definition I-6. A topological space X is said to be countably compact if every countable open covering of X contains a finite subcovering.

Definition I-7. A topological space X is said to be B-compact (Bolzano compact) if every infinite subset of X contains an accumulation point in X.

Theorem I-3. Every compact space is countably compact, and every countably compact space is B-compact.

Proof. The first assertion is clear. To prove the second, let X be a countably compact space. Let A be an infinite subset of X.

It must be shown that X contains an accumulation point of A. Assume that X contains no accumulation point of A. Then there exists a sequence $\{x_i\}$ of distinct points of A having no limit in X. Define $B = \{x_i\}$. Now for each x_i there exists an open set O_i such that $O_i \cap B = x_i$. If this were not the case, then x_i would be an accumulation point of A in X. B is closed, since it contains no accumulation points; hence $O = X - B$ is open. Consider the collection $\{O_i ; O\}$. This is a countable collection of open sets which cover X. But by countable compactness there exists a finite subcovering of X. This fact is inconsistent with the countable infinity of the sequence $\{x_i\}$. From this contradiction it is seen that the assumption of A having no accumulation point in X is a false one. Hence X is B-compact.

Remark. We have as a corollary to theorem I-3 that every compact space is B-compact. The converse of this theorem in general is not true, as the following example shows.

Example I-1. A B-compact space which is not countably compact. Let K be the set of positive integers. Take as a basis of open sets the collection $V_n = \{2n, 2n - 1\}$. The set K with the basis $\{V_n\}$ is a topological space. K is not countably compact since the collection $\{V_n\}$ covers K, but contains no finite subcovering. It is B-compact since every nonempty subset of K contains an accumulation point. This is true since every even integer is an accumulation point of the set consisting of the preceding odd integer, and every odd integer is an accumulation point of the set consisting of the succeeding even integer. Every subset of K contains either even or odd integers.

<u>Theorem I-4</u>. If X is a topological space in which subsets consisting of one point are closed (T_1 spaces), and if X is B-compact, then X is countably compact.

<u>Proof</u>. Let X be a B-compact T_1 space. Let $\{O_n\}_{n=1}^{\infty}$ be a countable open covering of X, and assume that no finite subcovering exists. For every integer n, the set $K_n = X - \bigcup_{i=1}^{n} O_i$ is closed and nonempty. Let p_n be an arbitrary point of K_n and define $H = \bigcup_{n=1}^{\infty} p_n$. Suppose first that H is infinite. Since X is B-compact there is an accumulation point p of H in X. Now every open set containing p contains infinitely many points of H distinct from p. If this were not true, then some open set U exists containing p and containing only a finite number of points $\{x_1, x_2, \ldots, x_n\}$ of H distinct from p. Since it is assumed that points are closed subsets of X, the set $U - \bigcup_{i=1}^{n} x_i$ is an open set containing p which contains no points of H. This is clearly not possible if p is to be an accumulation point of H. Consequently p is an accumulation point of the set $H_n = \bigcup_{i=n}^{\infty} p_i$ for some n. But H_n is a subset of the closed set K_n, so $p \in K_n$ for every n. Hence $p \notin O_n$ for every n, contradicting the covering property of $\{O_n\}$. Now consider the case in which H is finite. In this case there exists $p \in H$, such that, given any integer N, there exists an $n > N$, such that $p_n = p$. Thus $p \in K_n$ for every n, which contradicts the covering property of $\{O_n\}$ as before. Since the assumption that no finite subcovering of $\{O_n\}$ for X leads to a contradiction, X is countably compact.

Example I-2. A countably compact space which is not compact. To construct the space, begin with an uncountable set S^* and well-order S^* with $<$. Consider the set $S_c = \{x : x \varepsilon S^*$ and there are uncountable many elements y of S^* such that $y < x\}$. If S_c is empty let $S = S^*$. If S_c is nonempty then there exists a least element Ω of S_c, since $<$ is a well-ordering. Define $S = \{x : x \varepsilon S^*, x < \Omega\}$. The set S so defined has the property that S is uncountable, but each $s \varepsilon S$ has countably many predecessors. Let the topological space S consist of the set S with the following basis for open sets: $I_{ab} = \{x : x \varepsilon S, a < x < b ; a \varepsilon S, b \varepsilon S\}$ together with the set $\{s_o\}$, where s_o is the least element of S. We first show that the space S is not compact. Consider the open covering $\left\{\{s_o\}, I_{s_o x} : x \varepsilon S\right\}$. If $\left\{\{s_o\}, I_{s_o x_k}\right\}_{k=1}^{n}$ were a finite subcovering, the maximum element $\bar{x}$ of $\{x_k\}_{k=1}^{n}$ would have only a countable number of predecessors. But S is uncountable, so there is an element x such that $\bar{x} < x$ and therefore is not covered by the finite subcollection. Hence S is not compact.

S is a T_1 space since clearly the complement of a single point is open. Therefore if S is B-compact, S is also countably compact. Let K be an infinite subset of S. Let a_1 be the least element of K and let a_n be the least element of $K - \bigcup_{i=1}^{n-1} a_i$ for $n \geq 2$. Define $K_o = \bigcup_{n=1}^{\infty} a_n$. We will show that $K_o \subset K$ has an accumulation point in S. K_o has an upper bound since otherwise $S = \left\{\{s_o\} \cup \bigcup_{n=1}^{\infty} I_{s_o a_n}\right\}$

would be countable, contradicting the uncountability of S. Therefore K_o has least upper bound a_o since the set of all upper bounds has a least element. For every element $a \varepsilon S$ such that $a < a_o$ there must exist an $x \varepsilon K$ such that $a < x < a_o$. Also there must exist an $a_n \varepsilon K_o$ such that $x < a_n < a_o$, or else x would be an upper bound of K_o preceding a_o. Therefore every open set I_{ab} containing a_o must contain an element of K_o. Hence a_o is an accumulation point of K_o and S is B-compact.

Definition I-8. A topological space X is a Lindelöf space if every open covering of X contains a countable subcovering.

Remark. Every countably compact Lindelöf space is compact.

Theorem I-5. Every topological space which has a countable basis of open sets (second countable spaces) is Lindelöf.

Proof. Let X be a second countable space. Let $\{O_a : a \varepsilon A\}$ be an open covering of X. A countable subcovering must be extracted. Let $\{U_i\}$ be a countable basis of X. For each x in X there exists an $a \varepsilon A$ such that $x \varepsilon O_a$. Also there exists a set U_i such that $x \varepsilon U_i \subset O_a$. Let $\{U_j\}$ denote the subcollection of $\{U_i\}$ with the property that for each U_j there exists an a such that $U_j \subset O_a$. For each j denote by O_j the element of $\{O_a\}$ which contains U_j. Clearly the collection $\{O_j\}$ is a countable subcovering of $\{O_a\}$ which covers X. Hence X is a Lindelöf space.

Corollary I-6. Every countably compact second countable space is compact.

Proof. Every second countable space is Lindelöf, and every countably compact Lindelöf space is compact.

<u>Definition I-9</u>. A topological space X is said to be sequentially compact if every sequence of elements of X has a subsequence which converges to a point in X.

<u>Theorem I-7</u>. Every sequentially compact space is countably compact.

<u>Proof</u>. Let X be a sequentially compact space. Let $\{O_i\}$ be a countable open covering of X. Define the set $K_n = X - \bigcup_{i=1}^{n} O_i$ for each n and notice that, if no finite subcovering of $\{O_i\}$ exists, K_n is nonempty for every n. From each K_n choose a point p_n and consider the sequence $\{p_n\}$. This sequence has a convergent subsequence $\{p_{n_i}\}$. Let p be the element in X to which $\{p_{n_i}\}$ converges. Let O_j be an open set in $\{O_i\}$ containing p. Then there exists an N such that for $i > N$, $p_{n_i} \varepsilon O_j$. Hence $p_{n_i} \notin K_j$. But $p_{n_i} \varepsilon K_{n_i}$ for some $i > N$. $n_i > j$ implies $K_{n_i} \subset K_j$. Therefore $p_{n_i} \varepsilon K_j$, but this is a contradiction.

<u>Remark</u>. An example of a space which is compact but not sequentially compact is given later in this chapter.

<u>Theorem I-8</u>. If the topological space X has a countable basis of open sets at each point (first countable spaces), and if X is B-compact, then X is sequentially compact.

<u>Proof</u>. Let $\{x_n\}$ be a sequence of elements in X. Then the set $\{x_n\}$ may be assumed to be an infinite subset of X (if not, it clearly contains a convergent subsequence). Hence $\{x_n\}$ has an accumulation point p. Let $\{V_n\}$ be a countable base at p. Define the

collection $\{U_n\}$ by $U_1 = V_1$ and $U_n = U_{n-1} \cap V_n$. $\{U_n\}$ is also a countable base at p which is linearly ordered by inclusion. Define the sequence $\{y_n\}$ by choosing $y_i \in U_i$ with $y_i \in \{x_n\}$ and such that if $y_i = x_{n_1}$ and $y_{i-1} = x_{n_2}$ then $n_1 > n_2$. $\{y_i\}$ is a subsequence of $\{x_n\}$, and $\{y_n\}$ converges to p. Hence X is sequentially compact.

C. Localization

Definition I-10. A topological space X is said to be locally compact if for each x in X, there exists a neighborhood of x whose closure is compact.

Theorem I-9. Every compact space is locally compact.

Proof. Let X be a compact space. Let x be a point in X and let U be any neighborhood of x. $\bar{U}$ is closed in X. Let $\{O_a\}$ be any open covering of $\bar{U}$. Then $\{O_a, X - \bar{U}\}$ is an open covering of X. X is a compact space and therefore $\{O_a, X - \bar{U}\}$ contains a finite subcovering of the form $\{O_{a_i}, X - \bar{U}\}$. Clearly $\{O_{a_i}\}$ covers $\bar{U}$.

Remark. It was actually shown in the above proof that every closed subset of a compact set is itself compact.

Theorem I-10. If X is locally compact and C is a compact subset of X, then there exists an open set U with the property that $U \supset C$ and $\bar{U}$ is compact.

Proof. Let $x \in C$. There exists an open set U_x containing x such that $\bar{U}_x$ is compact. The collection $\{U_x : x \in C\}$ covers C. Since C is compact, there exists a finite subcovering $\{U_{x_i}\}_{i=1}^{n}$

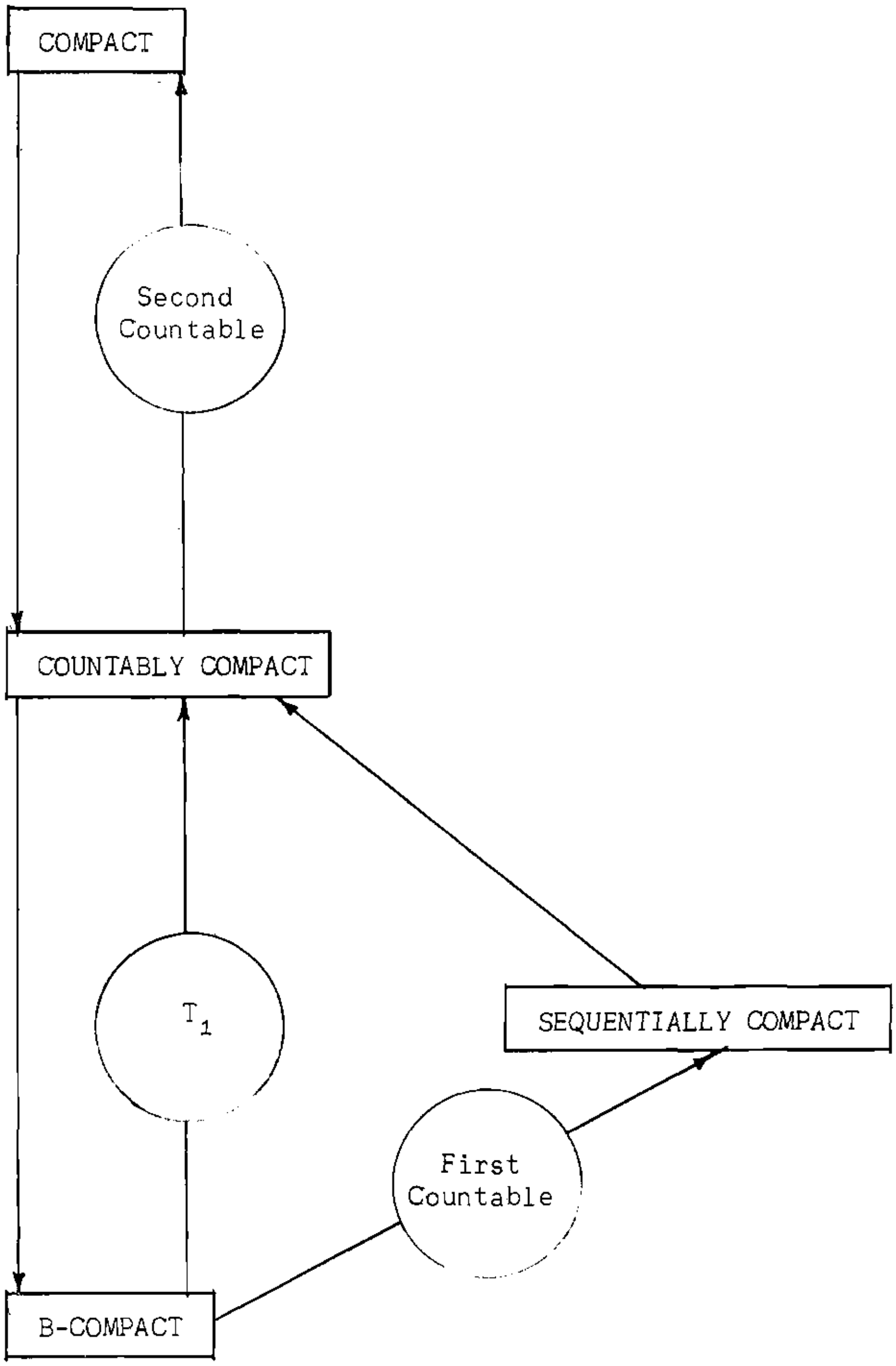

Figure 1. Relationship Between Types of Compactness

of C. Let $U = \bigcup_{i=1}^{n} U_{x_i}$. Then U is open and contains C. Furthermore $\overline{U} = \overline{\bigcup_{i=1}^{n} U_{x_i}} = \bigcup_{i=1}^{n} \overline{U}_{x_i}$ is compact.

D. Compactness and Hansdorff Spaces

Definition I-11. A topological space X is called Hausdorff if for each pair of distinct points x and y in X there exists two disjoint open sets U_x and U_y such that $x \in U_x$ and $y \in U_y$.

Theorem I-11. A compact subset of a Hausdorff space is closed.

Proof. Let X be a Hausdorff space and let C be a compact subset of X. It will be shown that $X - C$ is open. Let $p \in X - C$ and let y be any point of C. By the Hausdorff property, there exist open sets U_p and V_y containing p and y respectively such that U_p and V_y are disjoint. Consider the collection $\{V_y : y \in O\}$. This is an open covering of the compact set C, and hence there exists a finite subcovering $\{V_{y_i}\}$ of C. Let U_p^i be the open neighborhood of p corresponding to V_{y_i}. Then $U = \bigcap_{i=1}^{n} U_p^i$ is an open set containing p which is a subset of $X - C$. Hence $X - C$ is open and C is closed.

Remark. It will be useful to note also in the above proof that the set $V = \bigcap_{i=1}^{n} V_{y_i}$ is an open set containing C which does not intersect U.

Theorem I-12. The intersection of a closed set and a compact set is compact.

Proof. Let F be a closed set and let C be a compact set in some topological space X. Let $\{U_a : a \in A\}$ be an open covering of

$F \cap C$. The set $X - F$ is open and contains $C - F$. Hence $\{U_a ; X - F\}$ covers C. C is compact, so a finite subcovering $\{U_{a_i} ; X - F\}$ covers C. But $(X - F) \cap (F \cap C)$ is empty; hence $\{U_{a_i}\}$ covers $F \cap C$. Hence $F \cap C$ is compact.

Corollary I-13. In a Hausdorff space, the intersection of two compact sets is compact.

Proof. By theorem I-11 the two sets are closed, and theorem I-12 their intersection is compact.

Example I-3. A space in which the intersection of two compact sets is not compact. (By Theorem I-11 and Theorem I-12 the space cannot be Hausdorff and the compact sets cannot be closed.) Let R denote the space of real numbers with the usual topology, and let T be a space consisting of two abstract elements a and b with only the set T and the empty set as open sets. Let X be the Cartesian product of T and R, i.e., $X = T \times R$. Then X is the set of all ordered pairs of the form (a, x) or (b, y), where x and y are real numbers; and open sets are sets of the form

$$\{(a, x), \ (b, x) : x \in O, \ O \text{ is open in } R\} .$$

Let

$$C_1 = \{(a, x), \ (b, y) : x \in (0, 1) \text{ and } y \in [0, 1]\}$$

and let

$$C_2 = \{(a, x) : x \in [0, 1]\} .$$

It is easily seen that C_1 and C_2 are compact, but the intersection

$$C_1 \cap C_2 = \{(a, x) : x \in (0, 1)\}$$

is not compact.

Definition I-12. A topological space X is said to be regular if for each point $x \in X$ and each open set U containing x there exists an open set V containing x such that $\bar{V} \subset U$.

Theorem I-14. Every compact Hausdorff space is regular.

Proof. Let p be any point of X and let U be any open set containing p. $F = X - U$ is a closed set and is also compact. By the argument of the proof of Theorem I-11 and the Remark following that theorem, there exist open sets V and O containing p and F respectively such that V and O are disjoint. It follows that $p \in \bar{V} \subset U$.

E. Compactification

Definition I-13. Let X be any T_1 space and let w be any abstract element not in X. The space $X^* = X \cup \{w\}$, with a basis for a topology consisting of all open sets of X and all subsets U of X^* such that $X^* - U$ is a closed compact subset of X, is the one-point compactification of X.

Theorem I-15. The one-point compactification X^* of a space X is a compact space. Furthermore X^* is T_1.

Proof. First it will be shown that the collection of open sets for X^*, defined above, is a basis for a topology on X^*. It is sufficient to show that:

(1) Given $p \in X^*$, there exists a U in the basis such that $p \in U$,

(2) Given basis sets U and V, and any point p in $U \cap V$, there exists a member, W, of the basis such that $p \varepsilon W \subset U \cap V$.

If $p \varepsilon X$, clearly condition (1) is satisfied. Let $p = w$. Then the set $U = X^* - \{q\}$, where q is any point in X, contains w and $X^* - U = \{q\}$, which is a closed compact subset of X under the T_1 assumption. Now let U and V be any two members of the basis and assume that $p \varepsilon U \cap V$. If $p \varepsilon X$, (2) is clearly satisfied. Assume $p = w$. $U \cap V$ is open, since

$$(X^* - U) \cup (X^* - V) = X^* - U \cap V,$$

and the two sets on the left are closed compact subsets of X. If there are no members of X in $U \cap V$, $\{w\} = W$ is the desired set. If $q \varepsilon U \cap V \cap X$, the set $(X^* - U \cap V) \cup \{q\}$ is compact and closed. Hence let $W = (U \cap V) - \{q\}$. W is open and furthermore $w \varepsilon W \subset U \cap V$. X^*, with the indicated topology, is therefore a topological space. The complement of every set consisting of only one point is open, so X^* is a T_1 space.

Now it will be shown that X^* is compact. Let $\{U_a\}$ be any open covering of X^*. One of the elements, say U, of $\{U_a\}$ contains w. Hence $X^* - U$ is closed and compact, and consequently some finite subcollection $\{U_{a_i}\}$ of $\{U_a\}$ covers $X - U$. The finite subcovering $\{U_{a_i}; U\}$ covers X^*.

Remark. A space may be compactified by methods other than the one-point compactification. See [2] for a discussion and references regarding compactification.

F. Compactness in a Metric Space

Definition I-14. A topological space X is a metric space if there exists a real-valued function d defined on $X \times X$ such that

$$(1)\quad d(x, y) + d(y, z) \geq d(x, z)$$

$$(2)\quad d(x, y) = d(y, x)$$

$$(3)\quad d(x, y) \geq 0$$

$$(4)\quad d(x, y) = 0 \text{ if and only if } x = y,$$

and such that the collection of sets $\{N(y, \varepsilon) : y \varepsilon X, \varepsilon > 0\}$ where $N(y, \varepsilon) = \{x : d(x, y) < \varepsilon\}$ is a basis for the topology of X.

Definition I-15. A finite subset F of a metric space X is said to be an ε-net for X if for each x in X there exists a y in F such that $d(x, y) < \varepsilon$. A metric space is said to be totally bounded (or pre-compact) if it has an ε-net for every $\varepsilon > 0$.

Remark. Since the $\frac{1}{n}$-neighborhoods of a point in a metric space form a countable basis at that point, every metric space is first countable. Also if a metric space is separable (contains a countably dense subset), it is second countable, since the collection of countable neighborhoods of points of the countably dense subset is a basis for the topology of X.

Theorem I-16. Every countably compact metric space is totally bounded.

Proof. Assume that X is countably compact but not totally bounded. Then for some $\varepsilon > 0$, X does not contain an ε-net. Let $x_1 \varepsilon X$, and choose x_2 such that $d(x_1, x_2) \geq \varepsilon$. Now by induction define the sequences $\{x_n\}$ so that $d(x_n, x_i) \geq \varepsilon$ for $i = 1, 2, \ldots, (n - 1)$. Clearly the sequence $\{x_n\}$ contains no convergent subsequence. Hence X is not sequentially compact. But X is first countable. Therefore X is not countably compact. This contradiction proves the theorem.

Theorem I-17. Every countably compact metric space is separable.

Proof. Let X be a countably compact metric space. By the preceding theorem, X is totally bounded. For each integer n, let F_n be a finite subset of X which is a $\frac{1}{n}$-net for X. Define $F = \bigcup_{n=1}^{\infty} F_n$, and note that F is countable. Let x in X and $\delta > 0$ be given. Choose N sufficiently large to insure that $\frac{1}{N} < \delta$. There exists a $y \varepsilon F_N$ such that $d(x, y) < \frac{1}{N} < \delta$. Thus every δ-neighborhood of x contains a point of F. Hence F is dense in X.

Remark. The proof of Theorem I-17 also shows that every totally bounded metric space is separable.

Theorem I-18. A metric space is compact if and only if it is totally bounded and complete.

Proof. (1) Assume that X is compact. Then X is totally bounded by Theorem I-16. Let $\{x_n\}$ be a Cauchy sequence in X (i.e., for every $\varepsilon > 0$, there exists an N such that for n and m larger than N, $d(x_n, x_m) < \varepsilon$). By the Remark preceding Theorem I-16, X is first countable and thereby sequentially compact. Hence the sequence $\{x_n\}$ has a subsequence $\{x_{n_j}\}$ which converges to a point x in X. Now let N be a number large enough so that for $\varepsilon > 0$,

$$d(x, x_{n_j}) < \frac{\varepsilon}{2} \qquad \text{and}$$

$$d(x_m, x_{n_j}) < \frac{\varepsilon}{2} \text{ for all } m \text{ and } j \text{ larger than } N.$$

Then

$$d(x, x_m) \leq d(x, x_{n_j}) + d(x_m, x_{n_j}) < \varepsilon .$$

Therefore $\{x_n\}$ converges to x and X is complete.

(2) Assume that X is totally bounded and complete. It will be shown that X is sequentially compact. This will imply that X is compact by Theorem I-5, since X is second countable. Let $\{x_n\}$ be a sequence of elements in X and assume that no convergent subsequence exists. Set $\varepsilon_k = \frac{1}{k}$, for $k = 1, 2, \ldots$ and construct corresponding ε_k-nets, F_k, in X. At least one of the spheres $S_1 = N(y, 1)$, $y \in F_1$, contains an infinite subsequence of $\{x_n\}$. Let $\{x_n^1\}$ denote this subsequence. Now at least one of the spheres $S_2 = N(y, \frac{1}{2})$, $y \in F_2$, contains an infinite subsequence of $\{x_n^1\}$ which will be denoted by $\{x_n^2\}$. By induction define a sequence of sequences $\{x_n^1\}$, $\{x_n^2\}, \ldots, \{x_n^k\}, \ldots$ such that the subsequence $\{x_n^k\}$ is contained in one of the spheres $S_k = N(y, \frac{1}{k})$, $y \in F_k$. It will be shown that the diagonal sequence $\{x_n^n\}$ is a Cauchy sequence in X. Let $\varepsilon > 0$ be given. Choose a positive number N such that $\frac{1}{N} < \frac{\varepsilon}{2}$. Consider

integers n and m larger than N. Then $\{x_n^n\}$ and $\{x_m^m\}$ both lie in a sphere S_N. Hence for some $y \varepsilon F_N$,

$$d(x_n^n, x_m^m) \leq d(x_n^n, y) + d(x_m^m, y) .$$

Therefore

$$d(x_n^n, x_m^m) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

X is complete; hence $\{x_n^n\}$ converges to some limit in X. This proves that X is sequentially compact. Consequently X in compact.

<u>Theorem I-19</u>. (Heine-Borel) A subset of E_n (n-dimensional Euclidean space) is compact if and only if it is closed and bounded.

<u>Proof</u>. (1) Assume that S is a subset of E_n which is compact but not closed and bounded. In case S is not closed, there exists some sequence in S converging to some point not in S. In case S is not bounded, there exists a sequence in S with an infinite limit. Either case is a violation of B-compactness, and hence of compactness. S is therefore closed and bounded.

(2) Assume that S is closed and bounded. It will be shown that S is complete and totally bounded. Let $\{x_n\}$ be a Cauchy sequence in S. Then by the completeness of E_n, $\{x_n\}$ converges to some limit x in E_n. But x belongs to S since S is closed. It follows that S is complete.

Let R be an n-dimensional cube containing S. (Such a cube exists since S is bounded.) For each $\varepsilon > 0$, subdivide R into sub-cubes each of whose sides is of length $\varepsilon/\sqrt{n}$. The set of all vertices in R is finite. Let $x = (x_1, x_2, \ldots, x_n)$ be a point in S. Then

$$|x - v| = \sqrt{(x_1 - v_2)^2 + (x_2 - v_2)^2 + \cdots + (x_n - v_n)^2}$$

and

$$|x - v| \leq \sqrt{\left(\frac{\varepsilon}{\sqrt{n}}\right)^2 + \cdots + \left(\frac{\varepsilon}{\sqrt{n}}\right)^2} = \varepsilon,$$

where $v = (v_1, v_2, \ldots, v_n)$ is any vertex of a sub-cube containing x. For each v choose an $y \varepsilon S$ such that $|y - v| < \varepsilon$. The set of all such y is a 2ε-net for S. Hence S is totally bounded, and compact by Theorem I-18.

<u>Theorem I-20</u>. In a metric space the properties, compactness, countable compactness, B-compactness, and sequential compactness, are all equivalent.

<u>Proof</u>. By referring to Figure 1 it can be seen that the following must be shown:

(1) Every countably compact metric space is second countable.

(2) A metric space is T_1.

Statement (1) follows from Theorem I-16 and the Remark preceding Theorem I-16. To prove statement (2), let x be any point in a metric space. The complement of x can be expressed as the union of all neighborhoods of points of the space different from x which do not intersect x. This union is open. Hence the set $\{x\}$ is closed.

G. Products of Compact Spaces

Definition I-16. If X_a is a topological space corresponding to an element a in an indexing set A, the Cartesian product $\prod_{a \varepsilon A} X_a$ is the set of all functions defined on A such that $f(a) \varepsilon X_a$ for each a in A. Let F be some finite subset of A and let U_a be an open subset of X_a for every $a \varepsilon F$ with $U_a = X_a$ for every $a \varepsilon A - F$. A basis for the product topology of $\prod_{a \varepsilon A} X_a$ is the collection of all subsets of $\prod_{a \varepsilon A} X_a$ that are of the form $\prod_{a \varepsilon A} U_a$. $\prod_{a \varepsilon A} X_a$, together with the product topology, is called the "product space."

Theorem I-21. (Tychonoff) The Cartesian product of any collection of compact spaces is compact, relative to the product topology.

Proof. See [3] page 25, or [2] page 143.

Example I-4. A compact space which is not sequentially compact. Let $X = \prod_{a \varepsilon I} I_a$, where I_a is the unit interval $[0, 1]$ with the topology of E_1. Impose the product topology on X. A sequence of functions $\{f_n\}$, where $f_n : I \rightarrow I$, must be found so that the sequence contains no convergent subsequence. A sequence $\{f_n\}$ in X converges if and only if it is pointwise convergent in the usual sense on $[0, 1]$. To see this, first assume that the $\lim_{n \rightarrow \infty} f_n = f$, where f is in X. For each n choose $x_i \varepsilon [0, 1]$ for $i = 1, 2, \ldots, n$. Then a neighborhood containing f is a set $U = \bigcap_{i=1}^{n} \{g : |g(x_i) - f(x_i)| < \varepsilon\}$. Hence if for sufficiently large N, n larger than N implies that $f_n \varepsilon U$. Then

$$|f_n(x_i) - f(x_i)| < \varepsilon .$$

Any point in I may be included in the collection $\{x_i\}$; hence $\{f_n\}$ converges pointwise in the usual sense. Clearly if $\{f_n\}$ converges pointwise to f, $f_n \in U$ for any choice of $\{x_i\}_{i=1}^{n}$. Now a sequence of functions from I into I will be constructed which does not contain a pointwise convergent subsequence.

Let Q be the set of all sequences of positive integers. This set has power c; hence there exists a one-to-one function $h : I \to Q$. Let the sequence $\{f_n\}$ be defined as follows: For each x in I, consider $h(x) = \{k_n\}$, and let

$$f_i(x) = \begin{cases} 0, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd} \end{cases},$$

where n is the first integer such that $i = k_n$.

$$f_i(x) = 0, \quad \text{if } i \neq k_n \text{ for every } n .$$

Let $\{f_{n_r}\}$ be any subsequence of $\{f_n\}$. Then if $h^{-1}(\{n_r\}) = y$,

$$f_{n_r}(y) = \begin{cases} 0 \\ 1 \end{cases}, \text{ alternately.}$$

Hence $f_{n_r}(y)$ fails to converge, so that $\{f_{n_r}\}$ is not pointwise convergent on I. Hence $\{f_{n_r}\}$ cannot converge in the product space X. But X is the product of compact spaces and is compact by Theorem I-21. Therefore X is an example of a space which is compact, but not sequentially compact.

H. Continuous Transformations of Compact Spaces

Theorem I-22. The continuous image of a compact space is a compact space.

Proof. Let X be a compact space and let f be a continuous mapping of X onto Y. We wish to show that Y is compact. Consider an open covering $\{U_a\}$ of Y. The collection $\{f^{-1}(U_a)\}$ is an open covering of X, since f is continuous. Therefore there exists a finite subcovering $\{f^{-1}(U_{a_i})\}$ of X. Then $\{f[f^{-1}(U_{a_i})]\}$ is a finite subcovering of Y. Hence Y is compact.

Theorem I-23. The continuous image of a countably compact space is a countably compact space.

Proof. Clearly the above proof will be the same in the case of a countable collection $\{U_i\}$.

Theorem I-24. If X is B-compact and if f is a continuous one-to-one function and if $f(X) = Y$, then Y is B-compact.

Proof. Let A be an infinite subset of Y. Then $f^{-1}(A)$ is an infinite subset of X, and therefore has an accumulation point p in X. Consider a neighborhood U of $f(p)$. $f^{-1}(U)$ is a neighborhood of p. Hence there exists a point $x \neq p$ such that $x \varepsilon f^{-1}(A) \cap f^{-1}(U)$. But $f(x) \neq f(p)$, since f is one-to-one. Therefore $f(x)$ is a point of A in U different from p. Consequently $f(p)$ is an accumulation point of A.

Definition I-17. A mapping $f(X) = Y$ is said to be open (closed) if for each open (closed) set U in X, $f(U)$ is open (closed) in Y.

Theorem I-25. Let X be a locally compact space and let f be an open, continuous function from X onto a Hausdorff space Y. Then Y is locally compact.

Proof. Let y be a point of Y, and let $x \varepsilon f^{-1}(y)$. Let U be an open set containing x such that $\overline{U}$ is compact. Then since f is open and continuous $f(U)$ is an open set containing y and $f(\overline{U})$ is compact. Since Y is Hausdorff, $f(\overline{U})$ is closed, and by continuity $f(\overline{U}) \subset \overline{f(U)}$. But $f(\overline{U})$ contains $f(U)$; hence $f(\overline{U}) = \overline{f(U)}$. $f(U)$ is a neighborhood of y whose closure is compact. Hence Y is locally compact.

Example I-5. A B-compact space whose image under a continuous mapping is not B-compact.

Let X be the set of positive integers with the topology on the same as in Example I-1. Let Y be the set of positive integers with the topology in which every subset of Y is open (the so-called discrete topology). Let $f : X \rightarrow Y$ be defined by

$$f(2n) = n, \quad f(2n - 1) = n \; ; n = 1, 2, 3, \ldots$$

This function is continuous, since the inverse images of open sets are open. X is B-compact, as noted previously; but Y is not, since there are no accumulation points in Y.

Example I-6. A locally compact space whose image under a continuous function is Hausdorff but not locally compact.

First it will be shown that any space X with the discrete topology is locally compact. Let x be any point of X. Then $\{x\}$ is an open set containing X with a compact closure. Hence X is locally compact. Also any function f defined on X with range in any topological space is continuous. This is true since $f^{-1}(U)$ is open in X regardless of the nature of f and U. To give the desired example, we need only to display a space Y which is not locally compact. Then let X be the same set of points as Y with the discrete topology on X. Let the function f be the identity mapping, $f : X \to Y$, where $f(x) = x$ for each x in X.

Let Y be the set of all points in the Euclidean plane which are on the lines $y = 1/n$ and which are in the unit square. Also include the point $(1/2, 0)$. Let Y have the topology inherited from the plane. Y is not locally compact, for no neighborhood of $(1/2, 0)$ contains a compact closure. This is true since each neighborhood will contain at least one sequence of points which converge to a point on the real line different from $(1/2, 0)$. Hence Y is not locally compact and the example may be constructed.

Definition I-18. Let f be a function defined on an arbitrary topological space X with values in E_1 (one-dimensional Euclidean space). Then f is said to be upper semi-continuous at a point x in X, if for every $\varepsilon > 0$ there exists an open set U containing x such that

$$f(y) - f(x) < \varepsilon$$

for every y in U. f is said to be lower semi-continuous at x if for every $\varepsilon > 0$ there exists an open set U containing x such that

$$f(x) - f(y) < \varepsilon$$

for each y in U.

A function is said to be upper semi-continuous (or lower semi-continuous) on X if it has the property at each point.

Theorem I-26. Let X be a compact space. Let f be a real-valued function defined on all of X. If f is upper semi-continuous on X, then f attains a maximum value on X. If f is lower semi-continuous, f attains a minimum value on X.

Proof. The first conclusion will be proved. The proof of the second is analogous in an obvious way and will be omitted.

Let $A = \sup \{f(x) : x \in X\}$. It will be shown that an x_o in X exists such that $f(x_o) = A$. Suppose first that A is infinite. Then there exists a sequence of points $\{x_n\}$ of elements in X such that $\lim_{n \to \infty} f(x_n) = \infty$. The sequence $\{x_n\}$ is a net in X and since X is compact, there exists a subnet $\{x_{n_e} : e \in E\}$ converging to a point x_o in X. That is, $\{x_{n_e} : e \in E\}$ has the property that:

(a) for each positive integer i, there exists an e_o in E such that for $e \geq e_o$ $(e \in E)$, $n_e \geq i$;

(b) for each open set U containing x_o there exists an e' such that for $e \geq e'$, $x_{n_e} \in U$.

Let $\varepsilon > 0$ be given. Let U be a neighborhood of x_o such that $f(x) - f(x_o) < \varepsilon$ for $x \in U$. Let e_1 be a member of E such that $e \geq e_1$ implies that $x_{n_e} \in U$. Now there exists an integer N such that for $n \geq N$, $f(x_n) \geq f(x_o) + \varepsilon$, since $\lim_{n \to \infty} f(x_n) = \infty$. Let

e_2 be a member of E such that for $e \geq e_2$, $n_e \geq N$. Then for $e > e_1$ and $e \geq e_2$, $f(x_{n_e}) - f(x_o) < \varepsilon$ and $f(x_{n_e}) \geq f(x_o) + \varepsilon$. This contradiction proves that A must be finite.

Let y_n be a point of $f(X)$ such that $A - 1/n \leq y_n$ for each positive integer n. Let $x_n \varepsilon f^{-1}(y_n)$ for each n. The sequence $\{x_n\}$ is a net in X, and hence contains a subnet $\{x_{n_e} : e \varepsilon E\}$ converging to a point x_o in X. For each $\varepsilon > 0$ there exists an integer N such that $|A - f(x_n)| < \varepsilon$ for $n \geq N$. Also there exists an e_1 in E such that for $e \geq e_1$, $n_e \geq N$. Hence $|A - f(x_{n_e})| < \varepsilon$ for $e \geq e_1$. By the upper semi-continuity of f, there exists a neighborhood U of x_o such that $f(x) - f(x_o) < \varepsilon$ for $x \varepsilon U$. Also there exists an element e_2 of E such that $e \geq e_2$ implies that $x_{n_e} \varepsilon U$. For $e \geq e_1$ and $e \geq e_2$, $A - f(x_{n_e}) < \varepsilon$ and $f(x_{n_e}) - f(x_o) < \varepsilon$. Hence

$$A - \varepsilon < f(x_o) + \varepsilon \ .$$

Since this is true for every $\varepsilon > 0$, it follows that $A \leq f(x_o)$. But also $f(x_o) \leq A$. Hence $f(x_o) = A$.

<u>Theorem I-27</u>. Let X be a sequentially compact space. Let f be a real-valued function defined on all of X. If f is upper semi-continuous on X, then f attains a maximum value on X. If f is lower semi-continuous on X, f attains a minimum value on X.

<u>Proof</u>. The proof of this theorem is identical to the proof of the previous theorem, except a convergent subsequence is used in place of a convergent subnet.

CHAPTER II

COMPACT MAPPINGS

A. Compact Mappings and Equivalences

Definition II-1. Let f be a mapping (continuous function) from a topological space X onto a topological space Y. The mapping f is said to be compact if for each compact subset K of Y, $f^{-1}(K)$ is a compact subset of X.

Theorem II-1. If X is a compact space and if $f(X) = Y$ is a mapping with Y a Hausdorff space, then f is compact.

Proof. Let K be a compact subset of Y. Since Y is Hausdorff, K is closed. By the continuity of f, $f^{-1}(K)$ is closed. Since $f^{-1}(K)$ is a closed subset of a compact space, it is compact.

Remark. A set is said to be conditionally compact if its closure is compact.

Theorem II-2. Let $f(X) = Y$ be a mapping and assume that X and Y are Hausdorff spaces. Then f is compact if and only if the images of nonconditionally compact sets under f are nonconditionally compact.

Proof. (1) Suppose f is compact. Let K be a nonconditionally compact set in X. ($\overline{K}$ is not compact.) Suppose $\overline{f(K)}$ is compact. Then $f^{-1}[\overline{f(K)}]$ is compact and contains $\overline{K}$ since $f^{-1}[\overline{f(K)}]$ is closed. But $\overline{K}$ is a closed subset of $f^{-1}[\overline{f(K)}]$ and is hence compact. This contradiction shows the invariance of nonconditional compactness.

(2) Suppose that the invariance holds. Let $K \subset Y$ be compact and assume that $f^{-1}(K)$ is not compact. $f^{-1}(K)$ is closed since K

is closed. Hence $\overline{f^{-1}(K)} = f^{-1}(K)$ is not compact. But by the invariance of nonconditional compactness, $f[f^{-1}(K)] = K$ is not conditionally compact. This is false since $K = \overline{K}$. Hence f is compact.

Definition II-2. Let $f(X) = Y$ be a mapping. A subset A of X such that $A = f^{-1}[f(A)]$ is called an inverse set. If G is open in X, G_o will denote the set of all points x in G such that $f^{-1}[f(x)] \subset G$.

Remark. G_o is an inverse set.

Lemma II-3. If f is closed and if G is an open set in X, then G_o and $f(G_o)$ are open.

Proof. $G_o = \{f^{-1}(y) : f^{-1}(y) \subset G, \ y \in Y\}$. $X - G$ is closed; hence $Y - f(X - G)$ is open. It will be established that $f(G_o) = Y - f(X - G)$. Let $y \in f(G_o)$. Then $f^{-1}(y) \subset G$, and $f^{-1}(y) \cap (X - G)$ is empty. Since $f^{-1}(y)$ is an inverse set, $\{y\} \cap f(X - G)$ is empty and $y \in Y - f(X - G)$. This shows that $f(G_o) \subset Y - f(X - G)$. Now let $y \in Y - f(X - G)$. The inverse of y is completely contained in G. Hence $f^{-1}(y) \subset G_o$ and $y \in f(G_o)$. This establishes the equality. It follows that $f(G_o)$ is open. But $f^{-1}[f(G_o)] = G_o$, which is open by the continuity of f.

Theorem II-4. If $f(X) = Y$ is a closed mapping and if for each $y \in Y$, $f^{-1}(y)$ is compact, then f is compact.

Proof. Let K be a nonempty compact subset of Y and let β be an open covering of $f^{-1}(K)$. Let $y \in K$. A finite subcovering β_y of β covers $f^{-1}(y)$, since by hypothesis $f^{-1}(y)$ is compact. Let U^y be the union of all elements of β_y. U^y_o and $f(U^y_o)$ are open by

Lemma II-3. The collection

$$\{f(U_o^y) : y \varepsilon K\}$$

is an open covering of K which is reducible to a finite subcovering. Hence a finite subcovering from the collection $\{U_o^y : y \varepsilon K\}$ covers $f^{-1}(K)$ since $f^{-1}[f(U_o^y)] = U_o^y$. But each U_o^y is covered by the union of only a finite number of elements of β. Therefore a finite subcovering for $f^{-1}(K)$ may be extracted from β.

Theorem II-5. If $f(X) = Y$ is a compact mapping and Y is a locally compact Hausdorff space, then f is closed and point inverses are compact.

Proof. It is obvious that point inverses are compact. It will be shown that f is closed. Let F be a closed subset of X and suppose that $y \notin f(F)$ is an accumulation point of $f(F)$. Since Y is locally compact, an open set G exists such that $y \varepsilon G$ and $\bar{G}$ is compact. Now suppose $f(F) \cap \bar{G}$ were compact; then there exist open neighborhoods U and V of $f(F) \cap \bar{G}$ and y respectively such that $U \cap V$ is empty. This is true since Y is a Hausdorff space. But

$$(G \cap V) \cap f(F) \subset V \cap [\bar{G} \cap f(F)] \subset V \cap U = \emptyset .$$

However $(G \cap V) \cap f(F)$ is not empty since $G \cap V$ is a neighborhood of y. Hence $\bar{G} \cap f(F)$ is not compact. Since $\bar{G}$ is compact, $f^{-1}(\bar{G})$ is compact; and $F \cap f^{-1}(\bar{G})$ is compact by Theorem I-12. Also $f[F \cap f^{-1}(\bar{G})] = f(F) \cap \bar{G}$. Hence $f(F) \cap \bar{G}$ is compact. This contradiction proves the theorem.

<u>Corollary II-6</u>. Let f be a mapping from X onto Y and assume that Y is a locally compact Hausdorff space. A necessary and sufficient condition for f to be compact is that f be closed and that point inverses be compact.

<u>Proof</u>. The necessity follows from Theorem II-5, and the sufficiency follows from Theorem II-4.

<u>Theorem II-7</u>. Let $f(X) = Y$ be a compact mapping from a first countable space X to a first countable space Y. Then f is closed.

<u>Proof</u>. Let F be a closed subset of X. Let y be an accumulation point of $f(F)$ and let $\{y_i\}$ be a sequence of points in $f(F)$ which converge to y. Consider the compact set $K = \{y_i ; y\}$. Since f is compact $f^{-1}(K)$ is compact. Let x_i be chosen such that $x_i \varepsilon f^{-1}(y_i) \cap F$. The sequence $\{x_i\}$ contains a subsequence $\{x_{i_n}\}$ which converges to a point x in $f^{-1}(K)$. Then $f(x_{i_n}\} = y_{i_n}$ and by continuity $\lim_{n \to \infty} f(x_{i_n}) = f(x) = y$. Hence $y \varepsilon f(F)$ and $f(F)$ is closed.

<u>Corollary II-8</u>. If $f(X) = Y$ is a mapping from a first countable space X to a first countable space Y, then f is compact if and only if f is closed and point inverses are compact.

<u>Proof</u>. (1) Assume that f is compact. Then clearly point inverses are compact, and f is closed by Theorem II-7.

(2) Assume that f is closed and that point inverses are compact. Then f is compact by Theorem II-4.

<u>Theorem II-9</u>. Let $f(X) = Y$ be a compact mapping from a first countable space X to a first countable Hausdorff space Y. For

any $y \varepsilon Y$ and any open set U containing $f^{-1}(y)$, y is interior to $f(U)$.

<u>Proof</u>. Suppose that there exists an open set U containing $f^{-1}(y)$ for which $y \notin \text{int } f(U)$. Then there exists a sequence $\{y_i\}$ of elements of $Y - f(U)$ such that $\{y_i\}$ converges to y. The set $K = \{y_i ; y\}$ is compact and $f^{-1}(K)$ is compact. Let $x_i \varepsilon f^{-1}(y_i)$ for each i. $x_i \varepsilon U$ for each i and furthermore $\{x_i\}$ contains a convergent subsequence $\{x_{i_n}\}$. Let $\{x_{i_n}\}$ converge to x; then $f(x_{i_n}\}$ converges to $f(x)$ which must equal y. But $x \notin U$ and $f^{-1}(y) \subset U$. This is a contradiction which implies that $y \varepsilon \text{int } f(U)$.

<u>Remark</u>. Theorems II-1, II-2, II-7, and II-9 of this section may be found in [4]. Theorems II-4 and II-5 may be found in [5].

<u>Example II-1</u>. Every polynomial $P(z) = a_n z^n + \cdots + a_1 z + a_o$ defined on the complex plane is a compact mapping. To see this, consider a compact subset K of the complex plane. By Theorem I-19 K is closed and bounded, and it will be shown that $P^{-1}(K)$ is also closed and bounded. By the continuity of P, $P^{-1}(K)$ is closed. Assume that $P^{-1}(K)$ is not bounded and let $\{z_i\}$ be a sequence of points in $P^{-1}(K)$ which has an infinite limit. But

$$|P(z_i)| = |a_n z_i^n + \cdots + a_1 z_i + a_o|$$

$$= |z_i^n| \, |a_n + a_{n-1} z_i^{-1} + \cdots + a_o z_i^{-n}| .$$

As z_i approaches infinity, the term $|z_i^n|$ approaches infinity and the term

$$|a_n + a_{n-1} z^{-1} + \cdots + a_o z^{-n}|$$

approaches $|a_n|$. Hence $|P(z_i)|$ has an infinite limit. This contradicts the boundedness of K. Therefore $P^{-1}(K)$ is bounded, and compact by Theorem I-19.

Remark. The compact mapping from the complex plane into the complex plane has been studied by G. T. Whyburn and other writers as a generalization of the polynomial.

B. Light, Monotone, and Quasi-Open Mappings

Definition II-3. A mapping $f(X) = Y$ is said to be light if for every $y \in Y$, $f^{-1}(y)$ is totally disconnected (i.e., each component of $f^{-1}(y)$ consists of a single point).

Remark. A component is a maximal connected set.

Definition II-4. A mapping $f(X) = Y$ is said to be monotone if for each $y \in Y$, $f^{-1}(y)$ is a continuum (a compact, connected set).

Definition II-5. A mapping $f(X) = Y$ is said to be quasi-open if for each $y \in Y$ and any open set U containing a compact component of $f^{-1}(y)$, then y is interior to $f(U)$.

Theorem II-10. If $f(X) = Y$ is a compact monotone mapping from a first countable space X to a first countable Hausdorff space Y, then f is quasi-open. Furthermore if f is quasi-open and monotone and if X is locally compact, second countable, and Hausdorff with Y first countable and Hausdorff, then f is compact.

Proof. (1) Assume that f is compact and monotone. Let $y \in Y$. Since f is monotone, $f^{-1}(y)$ contains one and only one compact component. By Theorem II-9, for any open set U containing this compact component, $y \in \operatorname{int} f(U)$. Hence f is quasi-open.

(2) Assume that X is locally compact and that f is monotone and quasi-open. Let K be a compact subset of Y and let $\{x_i\}$ be a sequence of points in $f^{-1}(K)$. It will be shown that $\{x_i\}$ contains a convergent subsequence. Now the sequence $\{f(x_i)\}$ in K contains a subsequence converging to $y \in K$ and for convenience this subsequence will be denoted by $\{f(x_i)\}$. Let U be an open set containing $f^{-1}(y)$ such that $\bar{U}$ is compact. Such a set exists by Theorem I-10. Since f is quasi-open all but a finite number of the sets $f^{-1}[f(x_i)]$ intersect U, for otherwise the neighborhood $\operatorname{int} f(U)$ of y excludes an infinite number of $f(x_i)$. It will be shown that all but a finite number of $f^{-1}[f(x_i)]$ are contained in U. Suppose not. Then by the connectedness of each $f^{-1}[f(x_i)]$ there exists a sequence $\{z_n\}$ such that $z_n \in f^{-1}[f(x_i) \cap (\bar{U} - U)$. But $\bar{U} - U$ is compact; hence a subsequence $\{z_{n_j}\}$ converges to $z \in (\bar{U} - U)$. But $\{f(z_{n_j})\}$ converges to $y = f(z)$. Hence $z \in f^{-1}(y) \cap U$, and this is a contradiction. If all but a finite number of $f^{-1}[(x_i)]$ are contained in U, then all but a finite number of $\{x_i\}$ are contained in $\bar{U} \cap f^{-1}(K)$. Since by Theorem I-12 $\bar{U} \cap f^{-1}(K)$ is compact, some subsequence of $\{x_i\}$ is convergent to a point in $\bar{U} \cap f^{-1}(K)$. This proves that $f^{-1}(K)$ is compact.

<u>Corollary II-11</u>. Let $f(X) = Y$ be a monotone mapping from a locally compact, second countable, Hausdorff space X to a first countable Hausdorff space Y. Then f is compact if and only if f is quasi-open.

Proof. The corollary follows immediately from Theorem II-10.

Theorem II-12. If $f(X) = Y$ is a compact, monotone mapping from a first countable space X to a first countable space Y, then the inverse of a continuum in Y is a continuum in X.

Proof. Let K be a continuum in Y. Clearly $f^{-1}(K)$ is compact. It must be shown that $f^{-1}(K)$ is connected. Let $S_1 \cup S_2 = f^{-1}(K)$ be a separation for $f^{-1}(K)$. $f^{-1}[f(S_1)] = S_1$ since otherwise some $f^{-1}[f(x)]$ is separated. Hence $f(S_1 \cap \bar{S}_2) = f(S_1) \cap f(\bar{S}_2) = \emptyset$. But $f(\bar{S}_2) = \overline{f(S_2)}$ since f is closed. Hence $f(S_1) \cap \overline{f(S_2)} = \emptyset$. Likewise, $\overline{f(S_1)} \cap f(S_2) = \emptyset$ and also $K = f(S_1) \cup f(S_2)$. Hence $f(S_1) \cup f(S_2)$ is a separation for K, and this is a contradiction. Therefore $f^{-1}(K)$ is connected and is a continuum.

Remark. The hypothesis on X in Theorem II-12 is needed to insure that f is closed. By Theorem II-5, the hypothesis on X could be replaced by the requirement that Y be a locally compact Hausdorff space. Also a hypothesis that f be closed could be used.

Remark. Theorems II-10 and II-12 in this section may be found in [4].

C. Invariance of Local Compactness

Theorem II-13. If $f(X) = Y$ is a compact mapping from a locally compact first countable space X to a first countable Hausdorff space Y, then Y is locally compact.

Proof. Let $y \in Y$. It must be shown that there exists an open set containing y whose closure is compact. $f^{-1}(y)$ is compact and by

Theorem I-10 there exists an open set U containing $f^{-1}(y)$ such that $\overline{U}$ is compact. Also by Theorem II-9, $y \varepsilon \operatorname{int} f(U)$. Since $f(\overline{U})$ is a closed subset of Y and since f is continuous, $f(\overline{U}) = \overline{f(U)}$; and this set is compact. Hence $\operatorname{int} f(U)$ is an open set containing y whose closure is contained in $\overline{f(U)}$ and is therefore compact.

<u>Theorem II-14</u>. If X is locally compact, and if f is a quasi-open mapping such that $f(X) = Y$ with Y a Hausdorff space, and if for each $y \varepsilon Y$, $f^{-1}(y)$ contains a compact component, then Y is locally compact. (Such a mapping is called effectively quasi-open.)

<u>Proof</u>. Let D be a compact component of $f^{-1}(y)$. Then by Theorem I-10, there exists an open set U such that $\overline{U}$ is compact and $D \subset U$. Since f is quasi-open, $y \varepsilon \operatorname{int} f(U)$. Since f is continuous, $f(\overline{U}) \subset \overline{f(U)}$. Also since Y is Hausdorff and $f(\overline{U})$ is compact, $f(\overline{U})$ is a closed set containing $f(U)$. Hence $f(\overline{U}) = \overline{f(U)}$. $y \varepsilon \operatorname{int} f(\overline{U}) = \operatorname{int} \overline{f(U)}$ and $\overline{f(U)}$ is compact. It follows that Y is locally compact.

<u>Remark</u>. Theorem II-13 may be found in [4]. Theorem II-14 may be found in [5].

<u>D. Compact Mappings and Homeomorphisms</u>

<u>Definition II-6</u>. A mapping $f(X) = Y$ is called a homeomorphism if f is one-to-one and open. It is equivalent to say that f is one-to-one and f^{-1} is continuous.

<u>Theorem II-15</u>. If $f(X) = Y$ is a homeomorphism, then f is compact.

<u>Proof</u>. Let $K \subset Y$ be compact. Let β be an open covering of $f^{-1}(K)$. The collection $\{f(U) : U \varepsilon \beta\}$ is an open covering of K and hence has a finite subcovering $\{f(U_i) : U_i \varepsilon \beta,\ i = 1,2,\dots, n\}$. But $f^{-1}[f(U_i)] = U_i$ since f is one-to-one. It follows that $\{U_i\}$ covers $f^{-1}(K)$.

<u>Theorem II-16</u>. If $f(X) = Y$ is a compact, one-to-one mapping from a first countable space X to a first countable Hausdorff space Y, then f is a homeomorphism.

<u>Proof</u>. It will be shown that f is open. Let O be an open set in X and let $y \varepsilon f(O)$. $f^{-1}(y) \subset O$ since f is one-to-one. By Theorem II-9 $y \varepsilon \operatorname{int} f(O)$. Hence each point of $f(O)$ is interior to $f(O)$, and this implies that $f(O)$ is open.

<u>Theorem II-17</u>. Let $f(X) = Y$ be a compact mapping and suppose that K, a subset of X, is either closed or an inverse set. Then $f|_K$, i.e., f restricted to K, is compact.

<u>Proof</u>. Let C be a compact subset of $f(K)$. Now C is also a compact subset of Y; hence $f^{-1}(C)$ is a compact subset of X. If K is closed, $f^{-1}(C) \cap K$ is compact by Theorem I-12. If K is an inverse set $f^{-1}(C) \cap K = f^{-1}(C)$. But $f|_K^{-1}(C) = f^{-1}(C) \cap K$ in any case. Hence $f|_K$ is compact under either of the prescribed conditions.

<u>Corollary II-18</u>. Let $f(X) = Y$ be a compact mapping from a first countable space X to a second countable Hausdorff space Y.

Let K be the set of points in X for which f is one-to-one when restricted to K. Then $f|_K$ is a homeomorphism.

Proof. Clearly K is an inverse set; hence $f|_K$ is compact by Theorem II-17. Also $f|_K$ is a homeomorphism by Theorem II-16.

Theorem II-19. Let $f(X) = Y$ be an open mapping from a second countable Hausdorff space X to a space Y. Suppose further that to each y in Y, $f^{-1}(y)$ consists of exactly k points (i.e., f is a k-to-one mapping), then f is compact.

Proof. First it will be shown that f is a local homeomorphism; i.e., for each point x in X, there exists a neighborhood of x on which f is a homeomorphism. To show this, it is only necessary to show that there exists a neighborhood of x on which f is one-to-one. For a given x in X, consider the k points of $f^{-1}[f(x)]$. Construct k disjoint neighborhoods $U_1, U_2, \ldots, U_k$ such that each of the k points of $f^{-1}[f(x)]$ is in one and only one of the neighborhoods. Let $V = \bigcap_{i=1}^{k} f(U_i)$ and let $V_i = U_i \cap f^{-1}(V)$ for each i. Assume that $x \in V_1$; then it is claimed that f is one-to-one on V_1. If there were two points y and z in V_1 such that $f(y) = f(z)$, then $f^{-1}[f(y)]$ contains two points in V_1 and one point in each of the other $k - 1$ neighborhoods. Hence f is at least $(k + 1)$-to-one at $f(y)$. This contradicts the hypothesis and proves that f is a local homeomorphism.

Now it will be shown that f is compact. Note that Y is also second countable, since the collection $\{f(U_i)\}$ is a countable basis for Y, where $\{U_i\}$ is a countable basis for X. Let K be a compact subset of Y and let $\{x_i\}$ be a sequence of points in $f^{-1}(K)$. The sequence $\{f(x_i)\}$ contains a subsequence converging to a point y in K and assume for convenience that this subsequence is denoted by $\{f(x_i)\}$. The set $f^{-1}(y)$ is exactly k points and will be denoted by $\{z_1, z_2, \ldots, z_k\}$. Consider a linearly ordered, countable basis of neighborhoods of z_1. (That such a basis exists was shown in the proof of Theorem I-8.) Either there is at least one member of $\{x_i\}$ in each of these neighborhoods or else there is some neighborhood $V^1_{n_1}$ which contains no members of $\{x_i\}$. In the first case a subsequence of $\{x_i\}$ converges to z_1 in $f^{-1}(K)$ and the theorem is proved. If the second case holds, consider z_2 with the same process. Continue by induction until either a convergent subsequence is found or else the set $f^{-1}(y)$ is exhausted. But the second case cannot occur, since the set $f(\bigcup_{i=1}^{k} V^i_{n_i})$ would be an open neighborhood of y which contains no points of $\{f(x_i)\}$. Hence one of the points z_i is the limit of some subsequence of $\{x_i\}$.

E. The Factorization of Compact Mappings

Theorem II-19. If $f_1(X) = Y$ and $f_2(Y) = Z$ are compact mappings, then the composition $f = f_2 f_1$ is a compact mapping from X onto Z.

Proof. Let K be a compact subset of Z. Then $f_2{}^{-1}(K)$ is compact if f_2 is compact and $f_1^{-1}[f_2^{-1}(K)]$ is compact by the compactness of f_1. But $f_1^{-1}[f_2^{-1}(K)] = f^{-1}(K)$. Hence f is compact.

Definition II-7. Let $f(X) = Y$ be a mapping from a topological space X to a topological space Y. Two mappings $f_1(X) = M$ and

$f_2(M) = Y$ such that $f(x) = f_2[f_1(x)]$ for each x in X is called a factorization of f. The topological space M is called the middle space of the factorization.

Theorem I-20. Let $f(X) = Y$ be a compact mapping from a space X to a space Y. Let $f_2 f_1$ be any factorization of f such that the middle space $f_1(X) = M$ is Hausdorff. Then f_1 and f_2 are compact.

Proof. Let K be a compact subset of Y. Then since f is compact $f^{-1}(K)$ is compact. Furthermore $f_1[f^{-1}(K)]$ is compact. But $f_1[f^{-1}(K)] = f_1[f_1^{-1}[f_2^{-1}(K)]] = f_2^{-1}(K)$, so that $f_2^{-1}(K)$ is compact. Hence f_2 is compact. Let $C \subset M$ be compact. $f_2(C)$ is compact by continuity and $f^{-1}[f_2(C)]$ is compact by the compactness of f. Also $f^{-1}[f_2(C)] \supset f_1^{-1}(C)$. But $f_1^{-1}(C)$ is closed since C is closed in M. $f_1^{-1}(C)$ is then a closed subset of a compact set and is hence compact.

Theorem II-21. (Metrization Theorem) For every second countable, regular, T_1 space X there exists a metric defined on X which generates the topology of X.

Proof. See [1], page 122, or [2], page 125.

Lemma II-22. Let $f(X) = Y$ be a compact mapping from a separable metric space X to a Hausdorff space Y. If U is any open set of X with U_o the union of all components of $f^{-1}(y)$ contained in U for $y \in Y$, then U_o is open in X.

Proof. Assume the lemma is not true. Then for some $p \in U_o$, there exists a sequence of points $\{p_i\}$ in $X - U_o$ which converges to p. Let C_i be the component of $f^{-1}[f(p_i)]$ which contains p_i. Then $p \in \liminf (C_i)$, since each neighborhood of p contains all but a finite number of $\{p_i\}$ and hence intersects all but a finite number of $\{C_i\}$. $K = \{f(p_i) \; ; \; f(p)\}$ is compact and since f is compact, $\underline{f^{-1}(K)}$ is compact. Also $\bigcup_{i=1}^{\infty} C_i$ is contained in $f^{-1}(K)$ so that $\overline{\bigcup_{i=1}^{\infty} C_i}$ is compact. Hence the conditions of Theorem (9.12) on page 12 of [6] are satisfied. From the conclusion of that theorem, lim sup (C_i) -- the set of all points arbitrary neighborhoods of which intersect infinitely many C_i -- is connected. For each element x in lim sup (C_i) a subsequence $\{x_{i_j}\}$ exists such that $x_{i_j} \varepsilon C_{i_j}$ and $\{x_{i_j}\}$ converges to x. By continuity, $f(x) = f(p)$; hence lim sup$(C_i) \subset f^{-1}[f(p)]$. Since $p_i \notin U_o$, $C_i \not\subset U$. A sequence $\{x_i\}$ such that $x_i \varepsilon C_i \cap (X - U)$ may be chosen, and by the compactness of $f^{-1}(K)$ some subsequence of $\{x_i\}$ converges to a point x in lim sup (C_i). But $x \notin U$ so that lim sup $(C_i) \not\subset U$. Consequently the component of $f^{-1}[f(p)]$ containing p is not contained in U. But this contradicts $p \varepsilon U_o$. The contradiction proves the lemma.

<u>Theorem II-23</u>. Any compact mapping $f(x) = Y$ from a separable metric space X to a Hausdorff space Y admits a factorization $f = f_2 f_1$, where f_1 is compact and monotone and f_2 is compact and light.

<u>Proof</u>. First the middle space M will be constructed. A point of M is defined to be a component of $f^{-1}(y)$ for some y in Y. A subset U of M is open if the same set of points as a subset of X is open. Clearly M with the prescribed open sets is a topological space. It will be shown that M is a regular, second countable T_1 space.

Let a be a point of M. If A is the corresponding set of points in X, then A is closed by the continuity of f. Hence $X - A$ is open, and the set C in M corresponding to $X - A$ is open. But $\{a\} = M - C$. Hence $\{a\}$ is a closed set, and M is T_1.

It will be verified that M is regular. Let $p \in M$ and let U' be an open set of M containing p. Let P and U be the corresponding sets in X. By the regularity of X, for each point $q \in P$ there exists a neighborhood V_q of q such that $q \in V_q \subset \overline{V}_q \subset U$. The collection $\{V_q : q \in P\}$ is an open covering of the compact set P. Extract a finite subcovering $\{V_{q_i}\}$ and define $V = \bigcup_{i=1}^{n} V_{q_i}$. V has the property that $P \subset V \subset \overline{V} \subset U$. Let V_o be the set of all points in M whose correspondents are in V. Then by Lemma II-22 and the definition of open sets in M, V_o is open. Clearly $p \in V_o$. It will be shown that $\overline{V}_o \subset U'$. Let q be a point of $M - U'$ and let Q be the corresponding set of X. Then $Q \subset X - U$. For each point $t \in Q$, there exists an open set containing t which does not intersect V, for $\overline{V}$ is a subset of U. Let W be the union of all such open sets for t varying over Q. W is an open set containing Q which fails to intersect V. Let W_o be the set of all points in M which have correspondents in X contained in W. W_o is an open set containing q which fails to intersect V_o. Hence no point of $X - U$ is an accumulation point of V_o. Therefore $\overline{V}_o \subset U'$, and M is regular.

To show that M is second countable, let $\{R_n\}$ be a countable basis for X. For any finite set of positive integers $n_1, n_2, \ldots, n_k$ define $R(n_1, n_2, \ldots, n_k) = \bigcup_{i=1}^{k} R_{n_i}$. Let $R_o(n_1, n_2, \ldots, n_k)$ be the subset of M consisting of all points of M whose correspondents are contained in $R(n_1, n_2, \ldots, n_k)$. Then by Lemma II-22 each $R_o(n_1, n_2, \ldots, n_k)$ is an open set in M. Let $p \varepsilon M$ and let U' be an open set containing p. Let P and U denote the corresponding sets in X. For each $t \varepsilon P$ there exists a member R_i^t of the countable basis such that $t \varepsilon R_i^t \subset U$. The collection $\{R_i^t : t \varepsilon P\}$ covers P, and since P is compact a finite subcovering exists. Let $R(i_1, i_2, \ldots, i_n) = \bigcup_{k=1}^{n} R_i^{t_k}$, where $\{R_i^{t_k}\}$ is the finite subcovering. Then $P \subset R(i_1, i_2, \ldots, i_n) \subset U$. Then $p \varepsilon R_o(i_1, i_2, \ldots, i_n) \subset U'$. Thus the sets R_o form a countable basis for M. The space M, then is a regular second countable T_1 space, and by Theroem II-21 M is a separable metric space.

For each $x \varepsilon X$, let $f_1(x)$ be the element of M whose correspondent contains x. For each $p \varepsilon M$, let $f_2(p) = f[f_1^{-1}(p)]$. By the definition of open sets in M, f_1 is continuous. Furthermore f_1 is closed. To see this, let K be a closed subset of X. Then $M - f_1(K)$ is the set of all elements y of M such that $f_1^{-1}(y)$ $X - K$. Hence $M - f_1(K)$ is open in M. Therefore $f_1(K)$ is closed and f_1 is closed.

Now it will be shown that f_2 is continuous. Let C be a closed subset of Y. Then $f^{-1}(C)$ is closed in X and $f_1[f^{-1}(C)]$ is closed in M. But $f_2^{-1}(C) = f_1[f^{-1}(C)]$. Hence f_2 is continuous. For each y in M, $f_1^{-1}(y)$ is a compact component and is hence a continuum. Thus f_1 is monotone. Both f_1 and f_2 are compact by

Theorem II-20. Let $y \varepsilon Y$ and suppose that $f_2^{-1}(y)$ contains a component K consisting of more than a point. Since f_2 is compact, K is a continuum. Then $f_1^{-1}(K)$ is a continuum by Theorem II-12 and is hence contained in one and only one component of $f^{-1}(y)$. By definition of f_1, $f_1[f_1^{-1}(K)] = K$ is a single point. From this contradiction, it follows that f_2 is light.

Remark. The theorems of this section may be found in [4].

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